ON VARIATIONAL INEQUALITIES WITH MULTIVALUED OPERATORS WITH SEMI-BOUNDED VARIATION

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In this paper we explore some problems for the steady-state variational inequalities with multivalued operators (VIMO). As far as we know, in this variant the term VIMO had been first introduced in [1]. The results of this paper with respect to VIMO extend and/or improve analogous ones from [1-7]. We refused the regularity conditions of the monotonic disturbance of the multivalued mapping ([1]) and some other properties of the objects ([2]). Besides we considered a wider class of operators with respect to [3,4]. We are studying the connections between the class of radially semi-continuous operators with semi-bounded variation, the class of pseudo-monotone mappings, which is used earlier (for example, in [2]) on selector's language, and the class of monotone mappings. Moreover, for new class of operators the property of local boundedness is substituted for a weaker one with respect to [1,2,5,7,8]. Also we refuse the condition that $A(y)$ is a convex closet set owing to the forms of support functions.

Let X be a reflexive Banach space, X^* be its topological dual space, by $\langle\cdot,\cdot\rangle$ we denote the dual pairing on $X\times X^*$, 2^{X^*} be the totality of all nonempty subsets of the space X^*, $A: X\to 2^{X^*}$ be a multivalued mapping with $DomA=\{y\in X: A(y)\neq\emptyset\}$. $A: X\to 2^{X^*}$ is called *strong* iff $DomA=X$. Further for simplicity we will consider only strong mappings A. Let us consider the upper and lower support functions which are associated to A:

$$[A(y),\xi]_+=\sup_{d\in A(y)}\langle d,\xi\rangle,\qquad [A(y),\xi]_-=\inf_{d\in A(y)}\langle d,\xi\rangle,$$

and norms: $$[\![Ay]\!]_+=\sup_{d\in A(y)}\|d\|_{X^*},\qquad [\![Ay]\!]_-=\inf_{d\in A(y)}\|d\|_{X^*}.$$

We will consider the following VIMO

$$[A(y),\xi-y]_+ +\varphi(\xi)-\varphi(y)\ \geqslant\ \langle f,\xi-y\rangle\qquad \forall\xi\in\operatorname{dom}\varphi\cap K,\tag{1}$$

where f is a fixed element from X^* , $\varphi: X\to\overline{\mathbb{R}}=\mathbb{R}\bigcup\{+\infty\}$ is a proper convex lower semi-continuous function, $\operatorname{dom}\varphi=\{y\in X:\varphi(y)<\infty\}$, K is a convex weakly closed set.

Definition. $L: X\to\overline{\mathbb{R}}$ is called *lower semi-continuous*, if the following is satisfied: if $X\ni y_n\to y$ in X then $\varliminf_{n\to\infty} L(y_n)\geqslant L(y)$.

Definition[1]. Operator $A: X\to 2^{X^*}$ is called

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a) *radially semi-continuous*, if for each $y, \xi, h \in X$ the following inequality holds:
$$\varliminf_{t\to+0} [A(y+t\xi), h]_+ \geqslant [A(y), h]_-;$$
b) *operator with semi-bounded variation*, if for each $R > 0$ and arbitrary $y_1, y_2 \in X$ such that $\|y_i\|_X \leqslant R$ $(i = 1, 2)$ the following inequality holds:
$$[A(y_1), y_1 - y_2]_- \geqslant [A(y_2), y_1 - y_2]_+ - C(R; \|y_1 - y_2\|'_X),$$
where $C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous, $\tau^{-1}C(r_1, \tau r_2) \to 0$ as $\tau \downarrow 0$ for each $r_1, r_2 > 0$, $\|\cdot\|'_X$ is a compact norm with respect to the initial norm $\|\cdot\|_X$;
c) *coercive operator*, if $\exists y_0 \in K$ such that
$$\|y\|_X^{-1}[A(y), y - y_0]_- \to \infty \text{ as } \|y\|_X \to \infty;$$
d) *locally bounded on* X, if for each $y \in X$ there exist $\varepsilon > 0$ and $M > 0$ such that $[\![A(\xi)]\!]_+ \leqslant M$ for each ξ such that $\|\xi - y\|_X \leqslant \varepsilon$;
e) *monotone*, if for each $y_1, y_2 \in X$ the following inequality holds:
$$[A(y_1), y_1 - y_2]_- \geqslant [A(y_2), y_1 - y_2]_+.$$
Definition[2]. Operator $A : X \to 2^{X^*}$ is called *pseudo-monotone*, if
i) the set $A(y)$ is nonempty, bounded, closed and convex at each $y \in X$;
ii) $A : F \to 2^{X^*}$ is locally bounded on each finite-dimensional subspace $F \subset X$;
iii) if $y_j \to y$ weakly in X, $w_j \in A(y_j)$ and $\varlimsup_{j\to\infty} \langle w_j, y_j - y\rangle \leqslant 0$, then for each element $v \in X$ there exists $w(v) \in A(y)$ with the property
$$\varliminf_{j\to\infty} \langle w_j, y_j - v\rangle \geqslant \langle w(v), y - v\rangle.$$
Definition[2]. Operator $A : X \to 2^{X^*}$ is called *generalized pseudo-monotone*, if from $y_j \to y$ weakly in X, $A(y_j) \ni w_j \to w$ $*$-weakly in X^* and $\varlimsup_{j\to\infty} \langle w_j, y_j - y\rangle \leqslant 0$ it follows that $w \in A(y)$ and $\langle w_j, y_j\rangle \to \langle w, y\rangle$.

Each pseudo-monotone operator is generalized pseudo-monotone one ([2]).

It is easy to see that each monotone operator is an operator with semi-bounded variation, the next result is connecting the classes of operators with semi-bounded variation and of pseudo-monotone operators. Simultaneously, we showed the interconnection between monotone and pseudo-monotone operators.
Definition. Operator $A : X \to 2^{X^*}$ is called *sequentially weakly locally bounded*, if for each $y \in X$ if $y_n \to y$ weakly in X then there exist a finite number N and a constant $M > 0$ such that $[\![A(y_n)]\!]_+ \leqslant M$ for each $n \geqslant N$.

Theorem 1. *Let A be a radially semi-continuous operator with semi-bounded variation. Then $\overline{co}A$ is pseudo-monotone, locally bounded and sequentially weakly locally bounded.*

Remark. It is enough to consider the weaker condition of the radially semi-continuity:
$$\varliminf_{t\to+0} [A(y+t\xi), -\xi]_+ \geqslant [A(y), -\xi]_-.$$
Let us consider solvability theorems.

Theorem 2. *Let K be a bounded convex weakly closed set and $A : X \to 2^{X^*}$ be a radially semi-continuous operator with semi-bounded variation. Then for each $f \in X^*$ the solution set of the inequality*
$$[A(y), \xi - y]_+ \geqslant \langle f, \xi - y\rangle \quad \forall \xi \in K \tag{2}$$
is nonempty and weakly compact in X. Moreover, there exists the element $w \in \overline{co}A(y)$ such that
$$\langle w, \xi - y\rangle \geqslant \langle f, \xi - y\rangle \quad \forall \xi \in K.$$

Proof. Let us consider the filter $\mathbb{F}$ of the finite-dimensional subspaces F of X. We construct the auxiliary operator $L_\varepsilon(\lambda, y) = \overline{co}\{(1-\lambda)P_\varepsilon(y)+\lambda(I_F^* f - I_F^* A(I_F y))\}$, where $I_F : X \to F$ is the inclusion map, $K_F = K \cap F$, $P_\varepsilon(y) = [K_F \cap (-N_{K_F}(y))] \setminus B_\varepsilon(y) - y$, $N_{K_F}(y)$ is the normal cone, $B_\varepsilon(y) = \{\xi \in K_F : \|\xi - y\|_F < \varepsilon\}$ and $\varepsilon \geqslant 0$ such that $K_F \setminus B_\varepsilon(y) \neq \emptyset$ for each y from ∂K_F. We can show that $L_\varepsilon(\lambda, \cdot)$ is upper semi-continuous on F. By construction for each $y \in \partial K_F$ we have that $L_\varepsilon(0, y) \cap T_{K_F}(y) \neq \emptyset$, where T_{K_F} is the tangential cone. If $\exists y \in \partial K_F$ and $\lambda \in [0, 1]$ such that $0 \in L_\varepsilon(\lambda, y)$ then this $y \in \partial K_F$ is a solution of VIMO on F. Else by Lere–Schauder theorem the inclusion $0 \in L_\varepsilon(0, y)$ has a solution on $intK_F$. Thus, we have the bounded sequence $\{y_F\} \subset K$. Using the generalized pseudo-monotonicity and the sequentially weakly locally boundedness of the operator $\overline{co}A$ we can find some limit element y which is a solution of (2). $\square$

Theorem 3. *Let the conditions of Theorem 2 be satisfied without K be a bounded set. If in this case operator A is coercive, then the statement of Theorem 2 holds.*

Proof. On each bounded set $K_R = K \cap B_R$ the solution y_R exists, by the coercivity of the operator $\overline{co}A$ under some R y_R is a solution of (2). $\square$

From Theorem 3 we can obtain following statement:

Theorem 4. *Let $A : X \to 2^{X^*}$ be a radially semi-continuous coercive operator with semi-bounded variation. Then $\forall f \in X^*$ the solution set of the inclusion*

$$\overline{co}A(y) \ni f$$

is nonempty and weakly compact in X.

Now we consider the based inequality (1) and the corresponding inclusion

$$\overline{co}A(y) + \partial\varphi(y) \ni f, \tag{3}$$

where $\partial\varphi(y)$ is the subdifferential of the function $\varphi : X \to \overline{\mathbb{R}}$ at the point $y \in X$.

Proposition. *Each solution of* (3) *satisfies VIMO* (1)*. If y is a solution of* (1) *and belong to $intK \cap \operatorname{dom}\partial\varphi$, then y is a solution of* (3).

This simple statement allows to study VIMO (1) using the inclusion (3).

Theorem 5. *Let $A : X \to 2^{X^*}$ be a radially semi-continuous operator with semi-bounded variation, $\varphi : X \to \overline{\mathbb{R}}$ be a proper convex lower semi-continuous function and the following coercivity condition satisfies:*

$$\exists y_0 \in \operatorname{dom}\varphi \cap K \textit{ such that}$$
$$\|y\|_X^{-1}\big([A(u, y), y - y_0]_- - \varphi(y)\big) \to +\infty \textit{ as } \|y\|_X \to \infty.$$

Then $\forall f \in X^*$ the solution set of (1) *is nonempty and weakly compact in X.*

Proof. Let us construct the auxiliary objects:

$$\widetilde{X} = X \times \mathbb{R}, \quad \widetilde{y} = (y, \mu) \in \widetilde{X}, \quad \widetilde{A}(\widetilde{y}) = (A(y), 0) \quad \forall \widetilde{y} \in \widetilde{X},$$
$$\widetilde{K} = \{(y, \mu) \in (K \cap \operatorname{dom}\varphi) \times \mathbb{R} \mid \mu \geqslant \varphi(y)\}, \quad \widetilde{f} = (f, -1),$$

We can prove that these objects satisfy all conditions of Theorem 2. Thus, the solution $\widetilde{y}$ exists and its first coordinate is a solution of (1). $\square$

Example. Let us consider the free boundary problem on Sobolev space $W_p^2(\Omega)$, $p \geqslant 2$:

$$-\sum_{i,j=1}^{n} a_{ij}(x, y, Dy)\frac{\partial^2 y}{\partial x_i \partial x_j} = f \text{ on } \Omega,$$
$$y \geqslant 0, \quad \frac{\partial y}{\partial \nu_A} \geqslant 0, \quad y\frac{\partial y}{\partial \nu_A} = 0 \quad \text{on } \Gamma,$$

where Ω is a sufficiently smooth simple connected domain of $\mathbb{R}^n$, Γ is the bound of Ω, the normal vector ν is defined at each $x \in \Gamma$, $\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{n} a_{ij}(x,y,Dy)\frac{\partial y}{\partial x_j} cos(x,\nu_i)$, $Dy = (\frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n})$. This problem can have not a classical solution, but we can find a weak solution on $W_p^2(\Omega)$. Let us assume that $a_{ij}(x,y,\xi)$ satisfy the following conditions:

(1) for each y, ξ the functions a_{ij} are continuous with respect to x,

(2) $\forall x \in \overline{\Omega}$ the functions a_{ij} are bounded with respect to ξ and y, and the following estimation holds: $|a_{ij}(x,y,\xi_1,\cdots,\xi_n)| \leqslant g(x) + k_0|y|^{p-2} + \sum_{i=1}^{n} k_i|\xi_i|^{p-2}$, where $k_i > 0$ $(i = \overline{1,n})$, if $p = 2$ then $g \in C(\Omega)$, and if $p > 2$ then $g \in L_{q'}(\Omega)$, $q' = p/(p-2)$,

(3) $\sum_{i,j=1}^{n} a_{ij}(x,y,\xi)\xi_i\xi_j \geqslant \gamma(R)R$, where $R = |y| + \sum_{i=1}^{n} |\xi_i|$ and $\gamma(R) \to +\infty$ as $R \to +\infty$.

Then the free boundary problem conforms to the following inequality:

$$[A(y), \xi - y]_+ = a_1(y, \xi - y) + [A_2(y), \xi - y]_+ \geqslant \langle f, \xi - y\rangle \quad \forall \xi \in K. \tag{4}$$

where $a_1(y,\xi) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x,y,Dy)\frac{\partial y}{\partial x_j}\frac{\partial \xi}{\partial x_i}dx$, $A_2(y) = \left(\frac{\partial}{\partial x_i}a_{ij}(x,y,Dy)\right)\frac{\partial \xi}{\partial x_j}$, $\frac{\partial a_{ij}}{\partial x_i}$ is the subdifferential of a_{ij}, $K = \{y \in W_p^2(\Omega) : y_{|\Gamma} \geqslant 0\}$ is a convex weakly closed set. We can prove that A is a radially semi-continuous coercive operator with semi-bounded variation. Thus, the inequality (4) has a solution.

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